Worst-case to Expander-case Reductions

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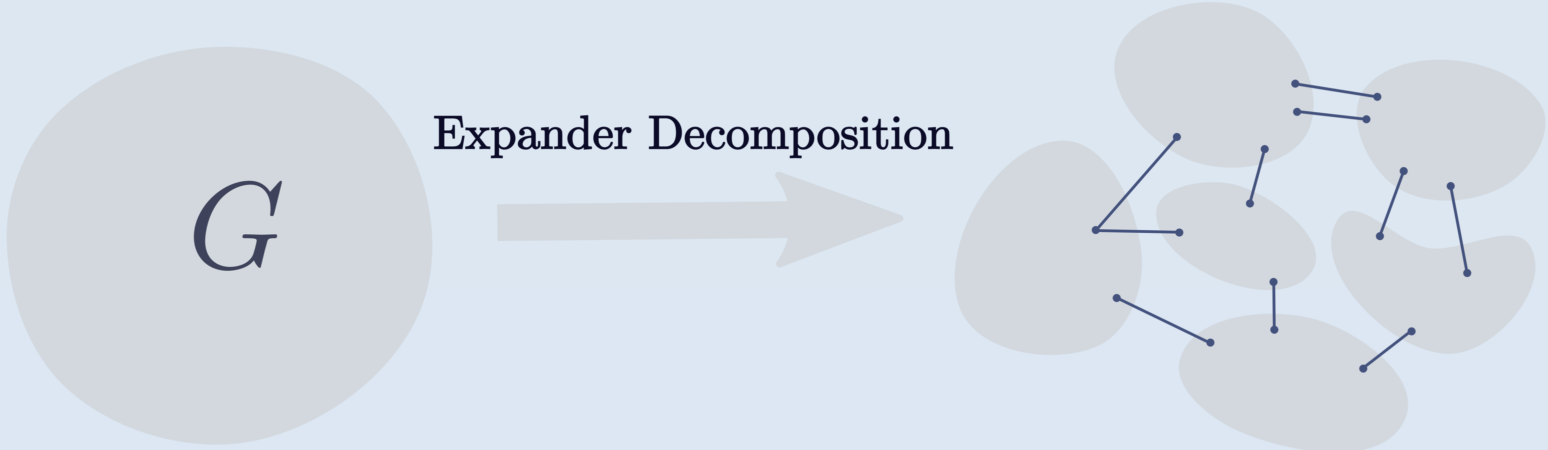
The Expander Decomposition Method:

- Choose some $\phi = (\log n)^{-\Omega(1)}$
- Decompose G into ϕ-expanders and $\tilde{O}(\phi m)$ outer-edges.
- Solve the problem on each ϕ-expander. Utilize the proprties of expanders to gain a speed-up!
- Combine the solutions to get a solution for G.



The expander decomposition method was used to achieve many breakthroughs in the recent years:

- Max-flow in $m^{1+o(1)}$ time.
- Dynamic shortest paths in $m^{1+o(1)}$ total time.
- Computing a Gomory-Hu tree in $n^{2+o(1)}$ time.
- Dynamic minimum spanning tree in $n^{o(1)}$ update.

⋮

Question 1: Are all problems as easy on worst-case graphs as on expanders?

Question 2: Is expander decomposition the right tool to prove this intuition?

Our main contribution: Direct Worst-case to Expander-case Reductions

G → **Reduction** → G'

Given input G, output G' such that:

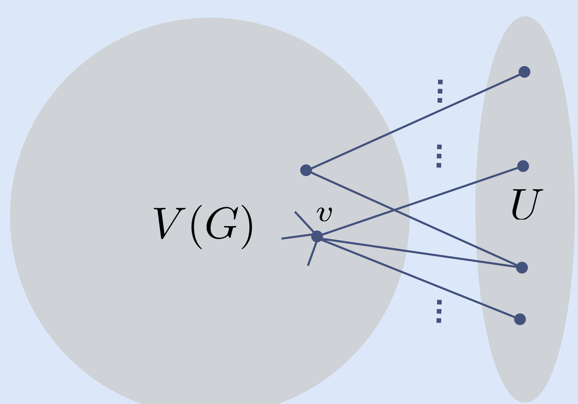
- G' is an $\Omega(1)$-expander.
- The solution for G can be easily computed from the solution for G'.
- The size of G' is near-linear in the size of G.

If a problem admits a Direct Reduction:

- **Positive answer for Question 1:**
  The problem is as easy on worst-case graphs as on expanders.
- **Negative answer for Question 2:**
  The expander decomposition method is useless against this problem, since we can always assume that the input graph is a good expander.

The basic expander construction:

- Add an expansion layer U of n vertices.
- From every $v \in V(G)$, sample $\deg(v)$ neighbors from U.

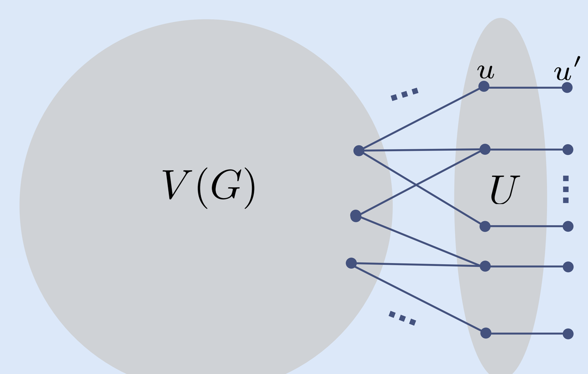


Claim: The resulting graph is an expander with high proability.

Direct Reduction for Maximum Matching:

Question: can expander decomposition help us resolve the complexity of Maximum Matching?

For every $u \in U$, add a degree-1 neighbor u'.



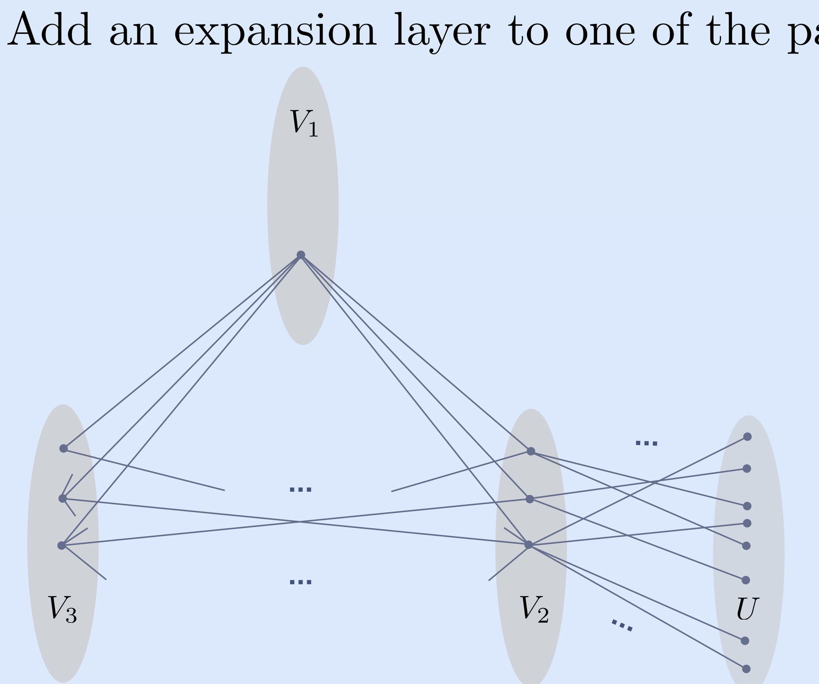
Claim: $MM(G') = MM(G) + n$

Answer: expander decomposition is useless against Maximum Matching!

Direct Reduction for Triangle Detection:

Question: can expander decomposition be used to refute the Triangle Detection conjecture?

- Make G tri-partite by taking three copies of $V(G)$.
- Add an expansion layer to one of the parts.



Answer: expander decomposition is useless against Triangle Detection!

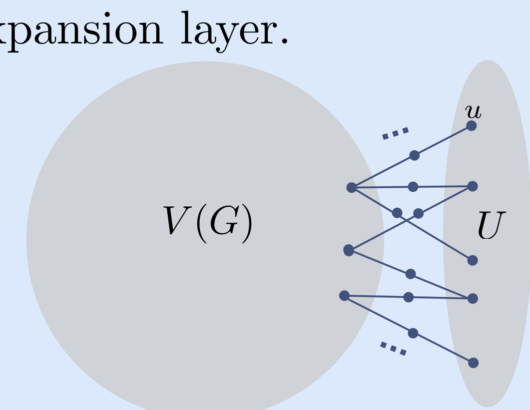
Additional problems that admit Direct Reductions:

- Minimum Vertex Cover
- Minimum Dominating Set

Direct Reduction for 4-Cycle Detection?

Question: can exander decomposition help us design any $n^{2-\Omega(1)}$ time algorithm for 4-Cycle Detection?

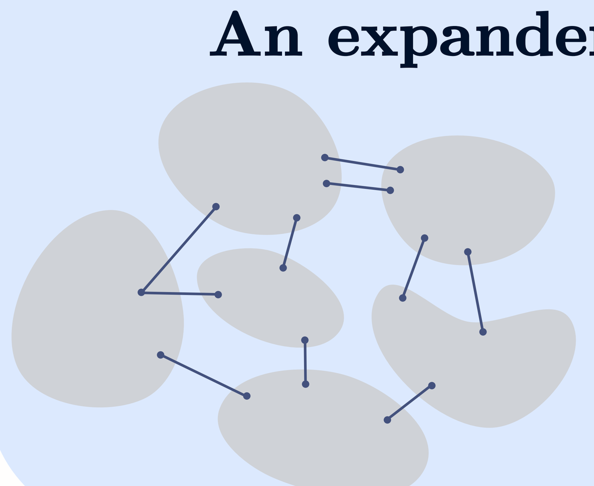
Naïve approach: subdivide the new edges to remove any 4-cycles that have been added by the expansion layer.



Caveat: the blowup in the number of vertices is huge: $|V(G')| = \Omega(m)$

Perhaps expander decomposition is the key?

An expander-decomposition based reduction:



Lemma: there is an algorithm that checks in $n^{2-\Omega(1)}$ time, whether there exists a 4-cycle that uses an outer-edge.

Almost gives a positive answer to Question 1, but not fully:

- The reduction runs in strongly super-linear time.
- ϕ is limited to $o(1)$.

Does not answer Question 2. Expander decomposition might be the key.