Equivalent Dichotomies for Triangle Detection in Subgraph, Induced, and Colored H-Free Graphs

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Proxy talk by: Shay Sapir

WEIZMANN INSTITUTE OF SCIENCE

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No other truly subcubic $n^{3-\Omega(1)}$-time algorithm is known!

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  APSP $\Rightarrow$ Minimum-Weight Triangle

- 3SUM in $n^{2-\Omega(1)}$ time?

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- Online Matrix-Vector Multiplication in $n^{3-\Omega(1)}$ time?

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FMM does not address weighted, listing, and online variants of Triangle Detection.

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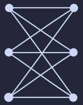

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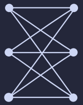

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For which patterns H can Triangle Detection in H-free graphs be solved by a combinatorial algorithm in $n^{3-\Omega(1)}$ time?

The Classification Objective

The Objective

For each pattern H, determine which of the following holds:

1. Triangle Detection in H-free graphs is solvable in $n^{3-\Omega(1)}$ time.
2. Triangle Detection in H-free graphs is as hard as Triangle Detection in general graphs.

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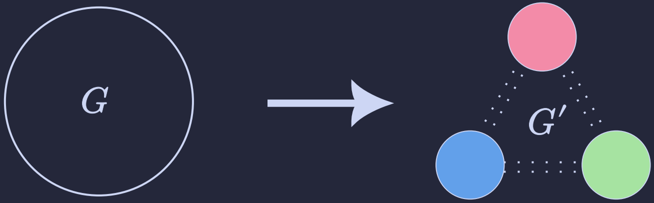
Lower Bounds: Color-Coding



Color the vertices randomly with 3 colors; delete monochromatic edges.

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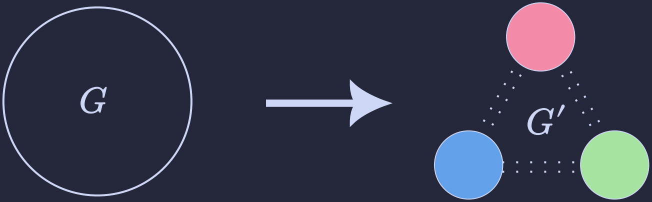

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- If G contains a triangle, it survives with probability $\Omega(1)$.
- Repeating this $O(\log n)$ times succeeds with high probability.
- Every produced subgraph is H-free for every pattern H with $\chi(H) > 3$.

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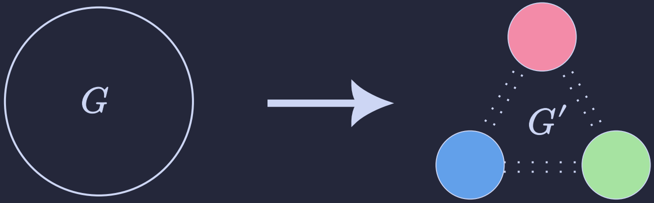

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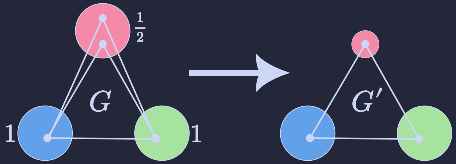
For $\chi(H) > 3$, Triangle Detection in H-free graphs remains as hard as Triangle Detection in general graphs.

Lower Bounds: The Case of Two or More Triangles in H

Idea: Well-known self-reduction to Unique Triangle Detection.



Random sieve: for each $i, j, k \in [\log_2 n]$, retain each red vertex with probability 2^{-i}, each blue vertex with probability 2^{-j}, and each green vertex with probability 2^{-k}.

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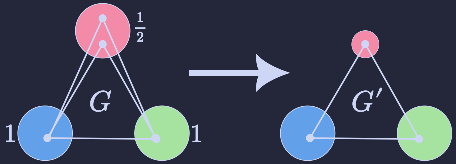

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- The number of produced subgraphs is $\mathrm{poly}(\log n)$.
- (NO-case) If G is triangle-free, then every produced graph is triangle-free and therefore H-free.
- (YES-case) If G contains a triangle, some produced graph G' has a unique triangle and therefore G' is H-free.

The Dichotomy Hypothesis

In our ITCS 2026 paper, we propose the following hypothesis:

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Triangle Detection in H-free graphs is classified as follows:

- Hard if H is not 3-colorable or contains at least two triangles; and
- Easy, otherwise.

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Results So Far

- For every pattern in the hard class, the problem has been confirmed to remain as hard as Triangle Detection in general graphs.
- An $n^{3-\Omega(1)}$-time algorithm is known only for a subset of the patterns in the easy class.

This Paper: Other Notions of Structure?

This investigation is also well-motivated in the induced setting.

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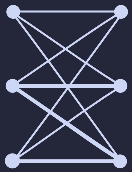

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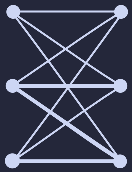

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Note: Induced H-free graphs are typically more complex.

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The following inclusions are straightforward:

$$\begin{array}{rcl} H\text{-free graphs} & \subseteq & \text{induced } H\text{-free graphs} \\ & \Downarrow & \\ \text{Easy(induced)} & \subseteq & \text{Easy(subgraph)} \\ \text{Hard(subgraph)} & \subseteq & \text{Hard(induced)} \end{array}$$

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A Priori Possibility

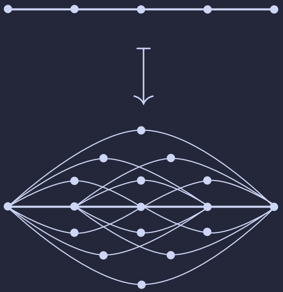
There could be a pattern H for which H-free graphs are easy, but induced H-free graphs are hard.

$$H \in \mathrm{Easy(subgraph)} \cap \mathrm{Hard(induced)}$$

Induced-to-Subgraph Reduction

Theorem
There is a mapping $H \longmapsto H^+$ preserving the "easy" properties:

$$\begin{aligned} H \text{ is 3-colorable} \quad &\Longrightarrow \quad H^+ \text{ is 3-colorable} \\ H \text{ has at most one triangle} \quad &\Longrightarrow \quad H^+ \text{ has at most one triangle} \end{aligned}$$



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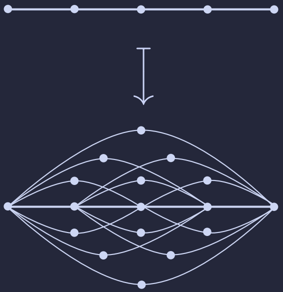

and an $n^{3-\Omega(1)}$-time reduction from Triangle Detection in induced H-free graphs to Triangle Detection in (subgraph) H^+-free graphs.

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In particular, assuming that the dichotomy hypothesis holds in the subgraph setting, we have:

$$\begin{aligned} \mathsf{Easy}(\text{subgraph}) &= \mathsf{Easy}(\text{induced}) \\ \mathsf{Hard}(\text{subgraph}) &= \mathsf{Hard}(\text{induced}) \end{aligned}$$

Takeaway: it is enough to focus on the (simpler) subgraph setting.

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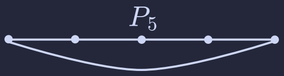

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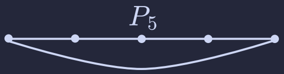
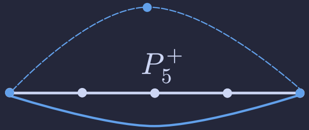


Idea: augment every non-edge with wedges.



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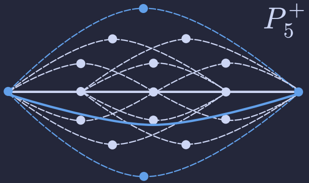

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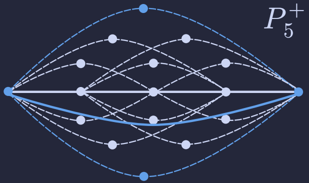

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Intuitively, the hardest instances have at most one triangle.

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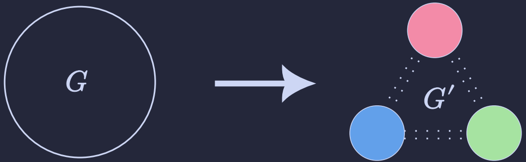
Recall: the first step in the reduction to Unique Triangle Detection is color-coding.



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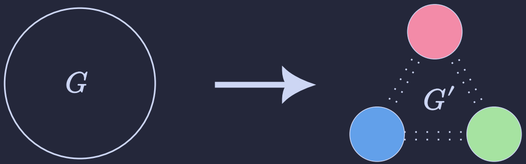
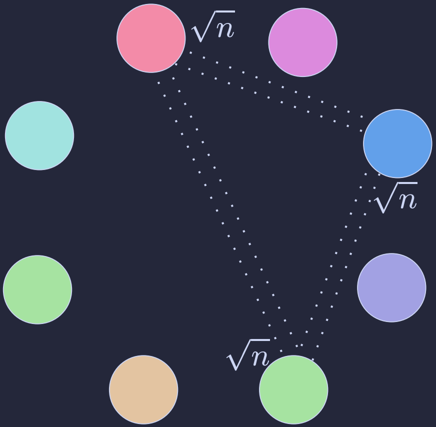


The Color-Coding Challenge
Randomly partitioning vertices and deleting monochromatic edges does not preserve induced H-freeness.

Color-Coding That Preserves Induced H-Freeness

Algorithmic Ramsey: Find either a triangle or an $O(\sqrt{n})$-coloring of G in linear time. Split the color classes into parts of size $O(\sqrt{n})$.



Every triple of color classes induces a 3-colored subgraph of G, and at least one such subgraph contains a triangle, if one exists.

Potential Follow-Ups

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