Recognizing Sumsets is NP-Complete

Amir Abboud, Nick Fischer, Ron Safier, *Nathan Wallheimer*



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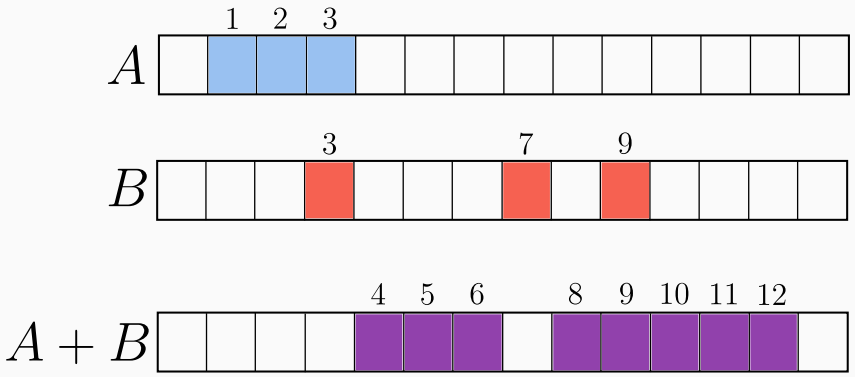
For $A, B \subseteq [0, M] := \{0, 1, \ldots, M\}$, let their sum be $A + B = \{a + b | a \in A, b \in B\}$.



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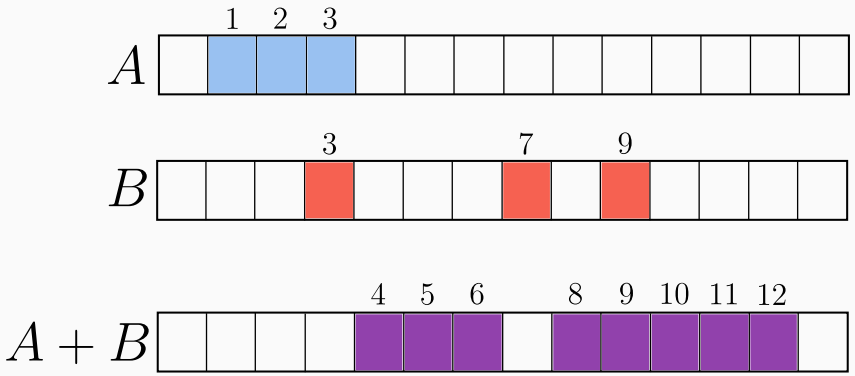

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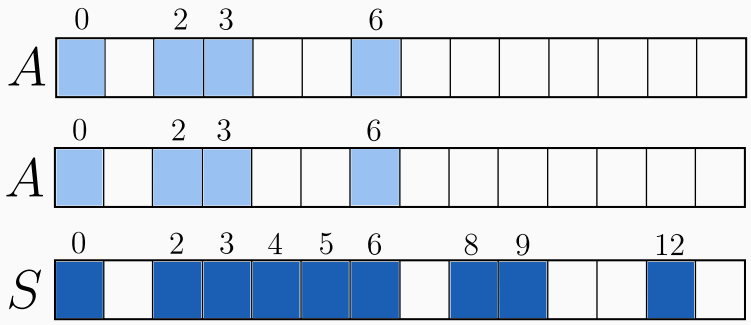
A set $S \subseteq [0, M]$ is called a *sumset* if $S = A + A$ for some $A \subseteq [0, M]$.



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Motivation: $|A + A|$ is a measure of structure in A.

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Example (Arithmetic Progressions)

If $A = [a, b]$, then $A + A = [2a, 2b]$, thus $|A + A| = 2|A| - 1$.
More generally: $|A + A| = 2|A| - 1 \iff A$ is an arithmetic progression.

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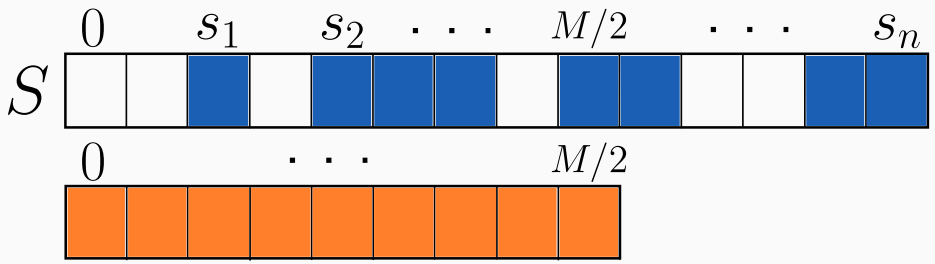
- Gain a better understanding of *factoring* problems.

Sumset Recognition Algorithms



An $O^\star(2^{M/2})$-time algorithm

Brute force over all subsets of $[0, M/2]$ and for every $A \subseteq [0, M/2]$, check if $A + A = S$.

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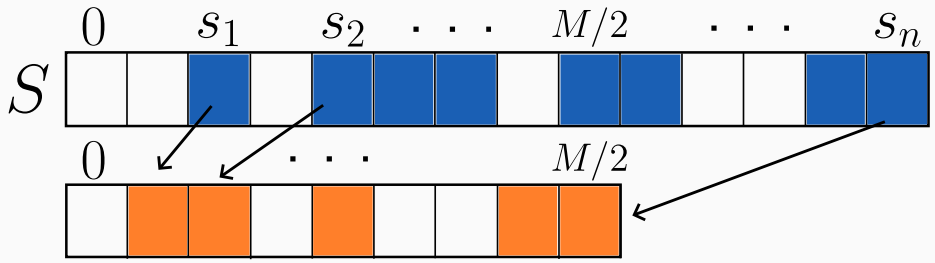

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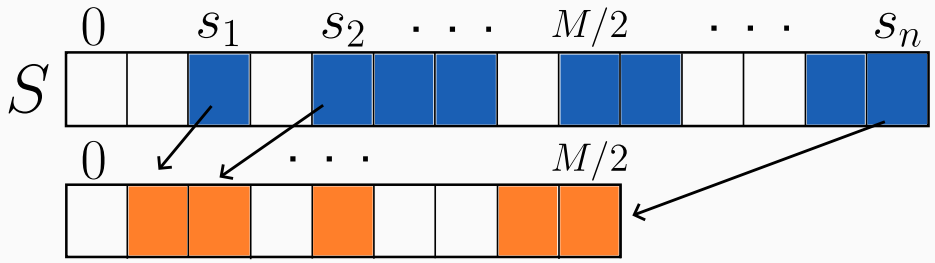

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Can we do better?

Our Contribution: Recognizing Sumsets is NP-Complete

Theorem (Reduction from 3-SAT)

There is a polynomial-time reduction that, given a 3-SAT formula ϕ with n variables, outputs a set $S \subseteq [0, O(n^4)]$, such that S is a sumset if and only if ϕ is satisfiable.

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Corollary (Exponential Time Hypothesis)

Assuming that solving 3-SAT requires $2^{\Omega(n)}$ time, then solving Sumset Recognition requires $2^{\Omega(M^{1/4})}$ time, and therefore also $2^{\Omega(n^{1/4})}$ time.

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Open Question: Resolve the gap between $2^{O(n)}$ and $2^{\Omega(n^{1/4})}$.

Proof Outline

We are given a 3-SAT formula ϕ with n variables and m clauses.
Goal: Design a set $S := S(\phi)$, and sets $\{A_\alpha\}_{\alpha \in \{0,1\}^n}$, such that:

$$\phi \text{ is satisfiable} \quad \Longleftrightarrow \quad S \text{ is a sumset.}$$

Moreover, $S = A_\alpha + A_\alpha$ for any satisfying assignment $\alpha \in \{0,1\}^n$.

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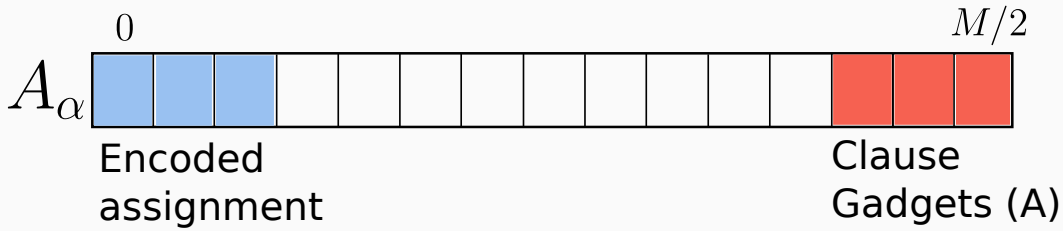

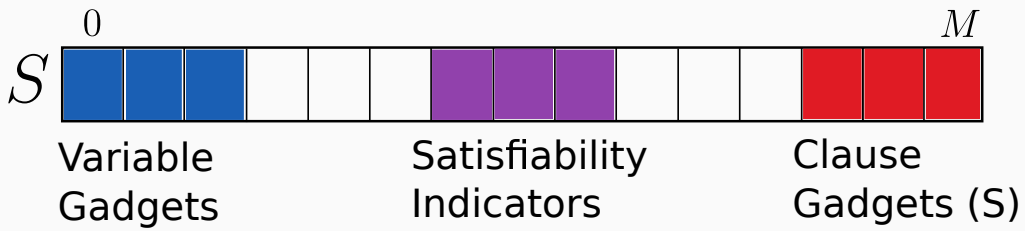

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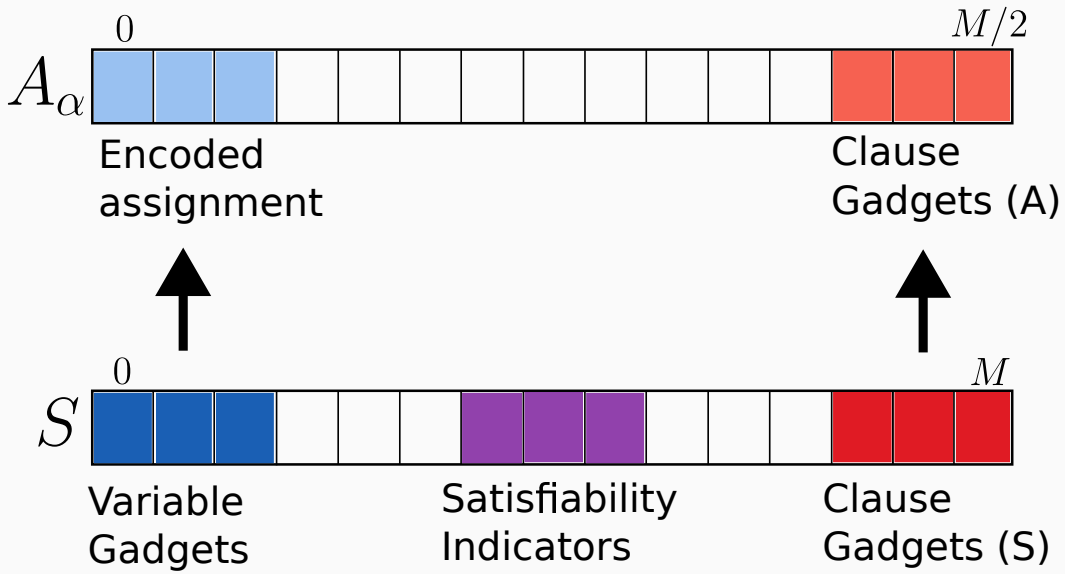

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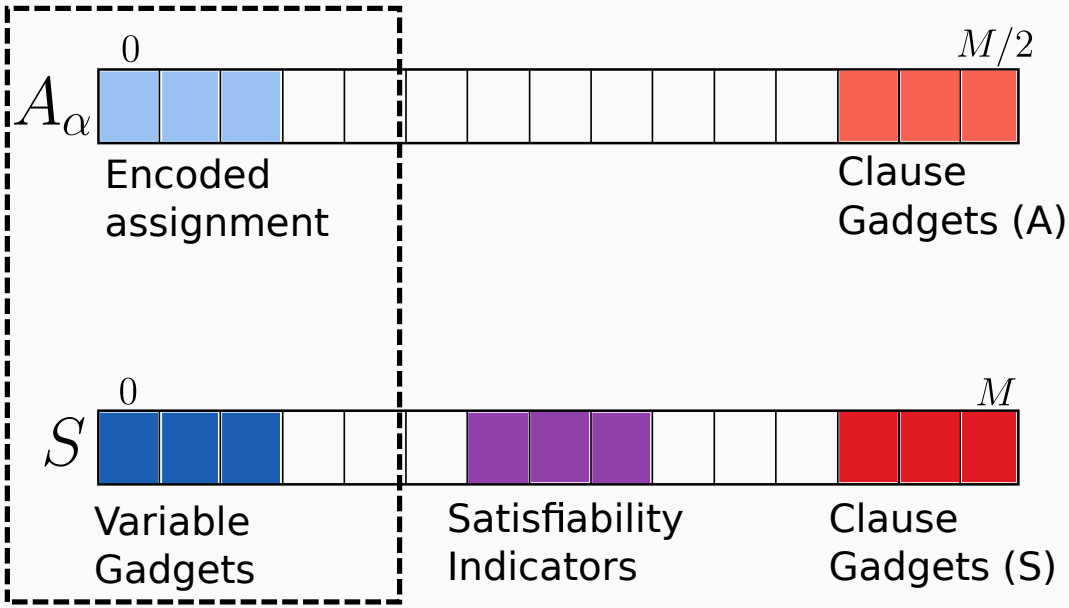
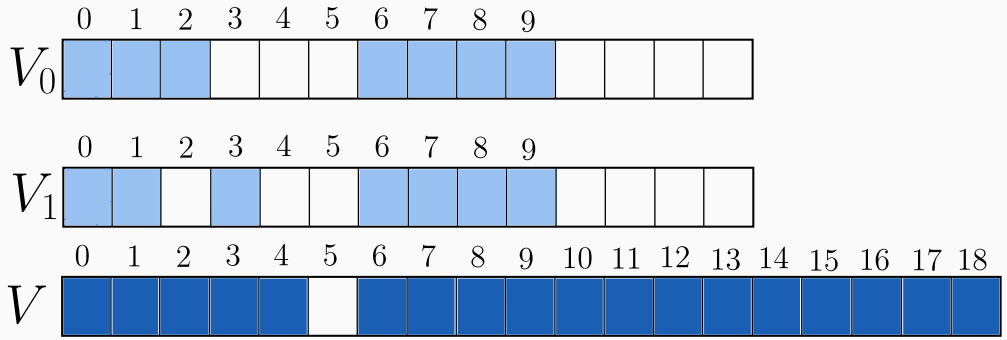


Variable Gadgets

Goal: Design a constant sized set V that has only two representations: $V = V_0 + V_0$ and $V = V_1 + V_1$.

$$V_0 = \{0, 1, 2, 6, 7, 8, 9\}$$
$$V_1 = \{0, 1, 3, 6, 7, 8, 9\}$$
$$V = [0, 18] \setminus \{5\}$$



Remark: This is the smallest variable gadget.

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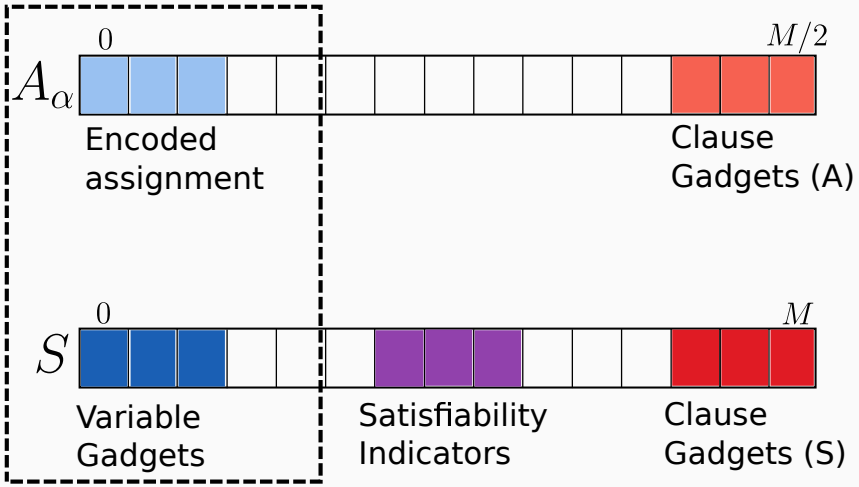

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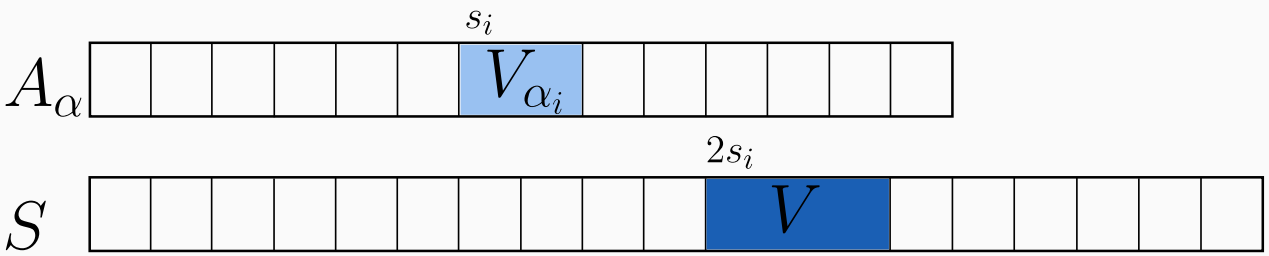

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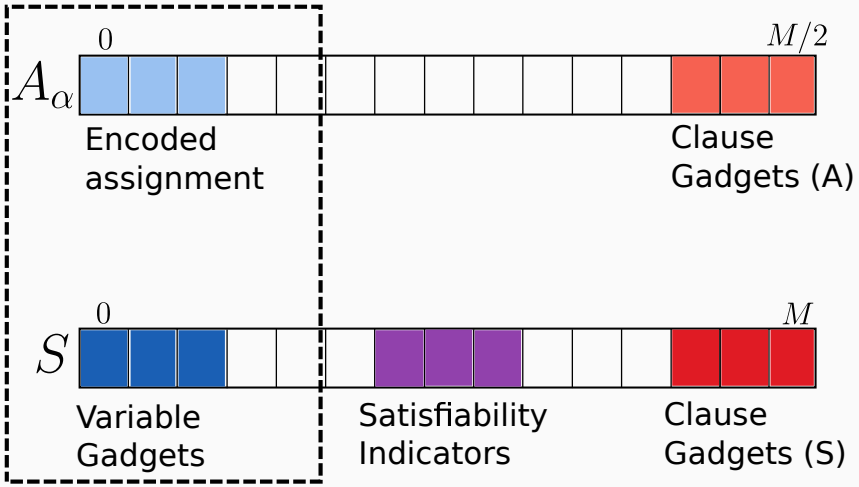

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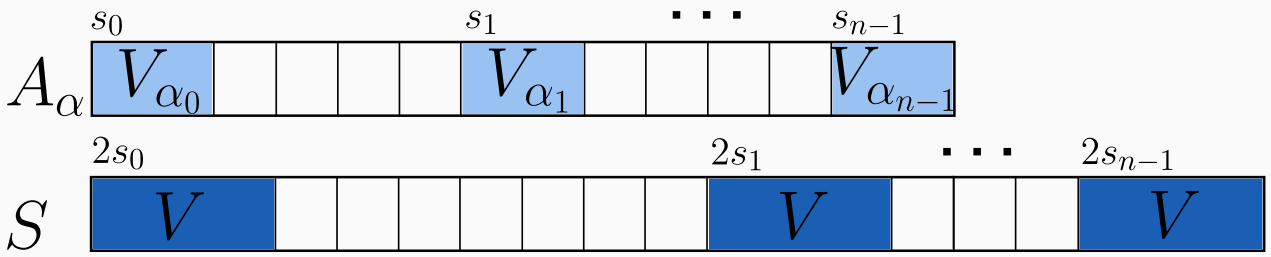

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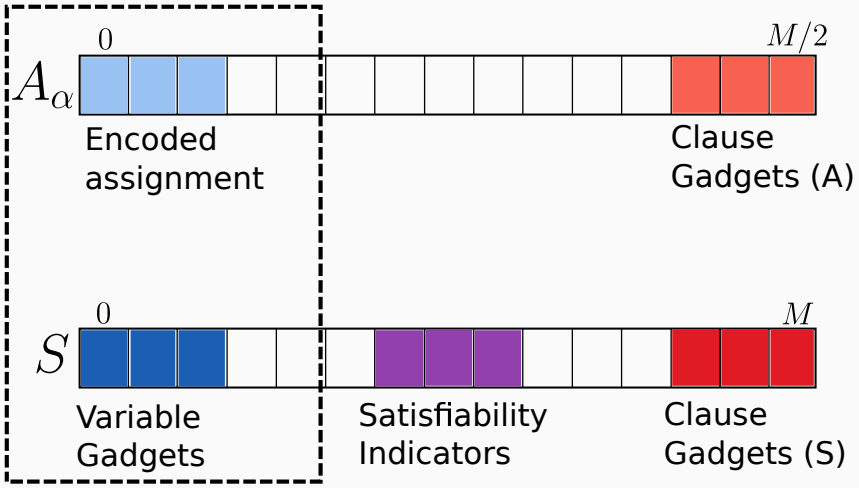

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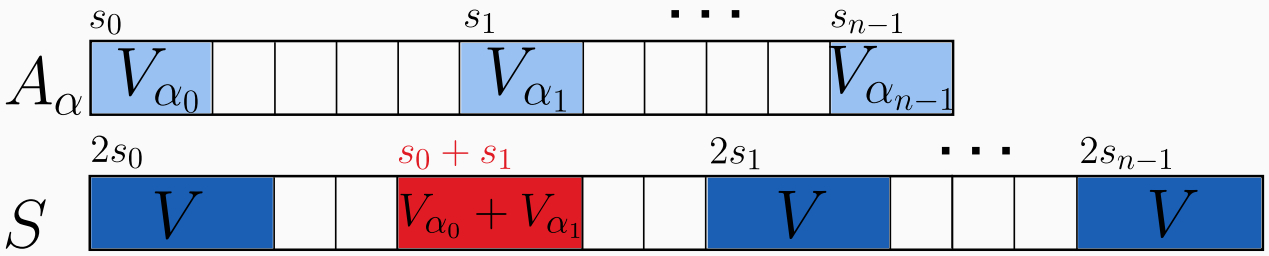
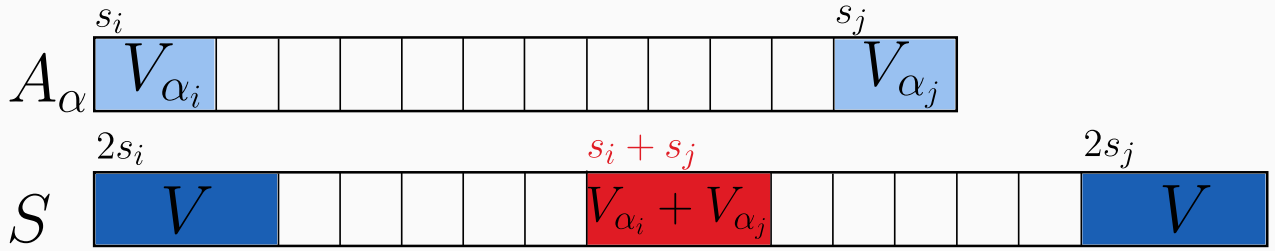


The cross-term $V_{\alpha_0} + V_{\alpha_1}$ is an unwanted by-product.

Cross-Terms



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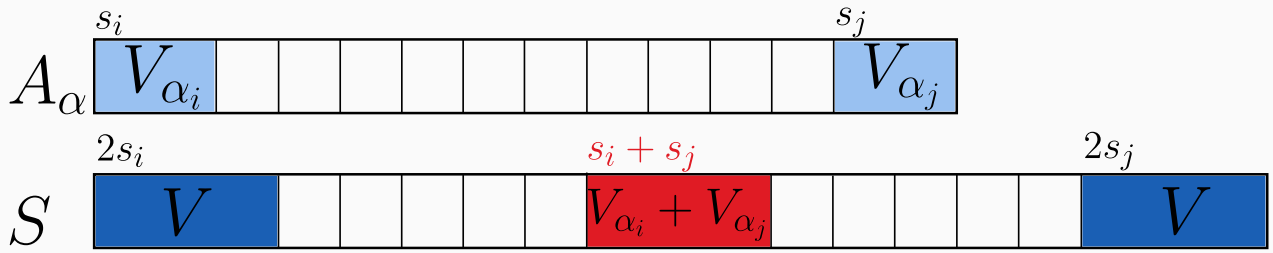

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Explicit Sidon Sets

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We get a Sidon set of size p over $[0, O(p^2)]$.

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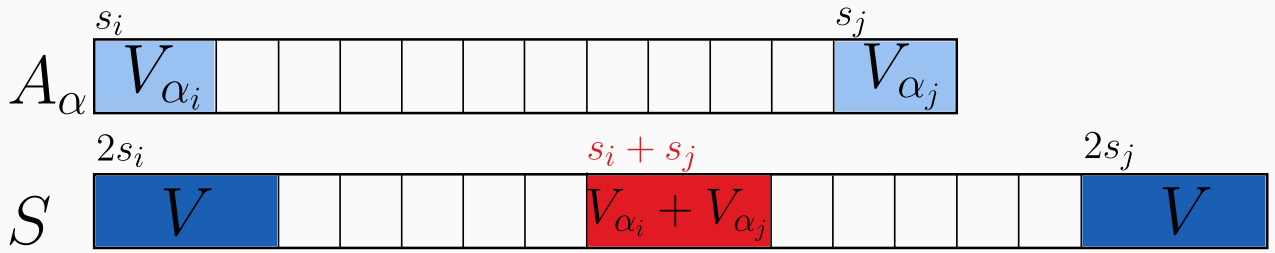

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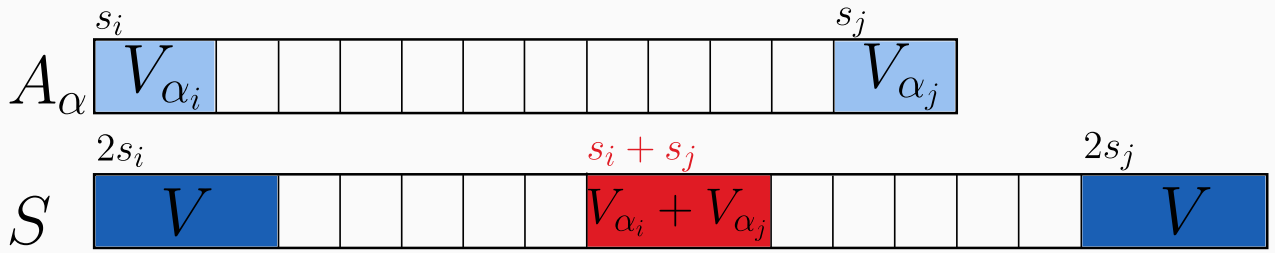

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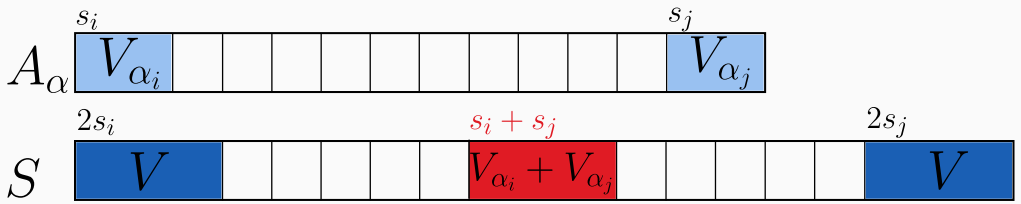
1. Having $V_{\alpha_i} + V_{\alpha_j}$ in S will make S dependent on α. The set S has to be *oblivious* to α.

Masking



Solution: Mask the cross-terms with complete intervals. Let $I = [\min(V), \max(V)]$ and $R = [\frac{\min(V)}{2}, \frac{\max(V)}{2}]$, so $R + R = I$.

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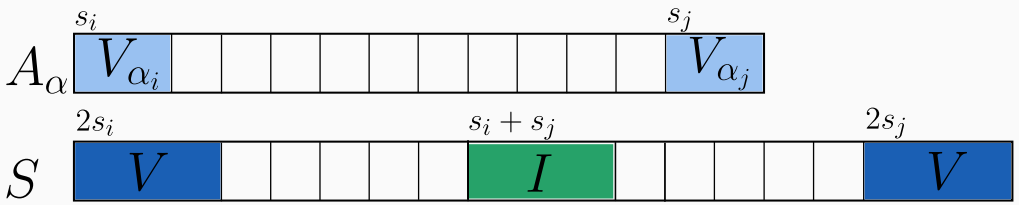

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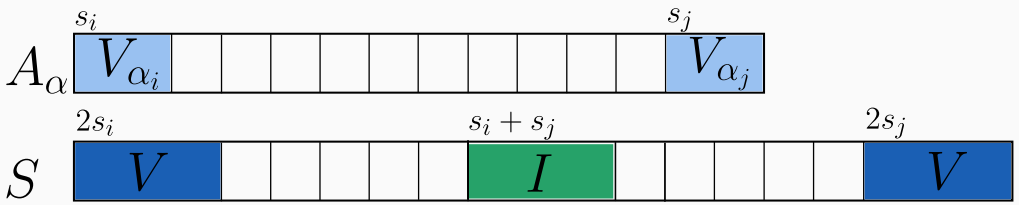

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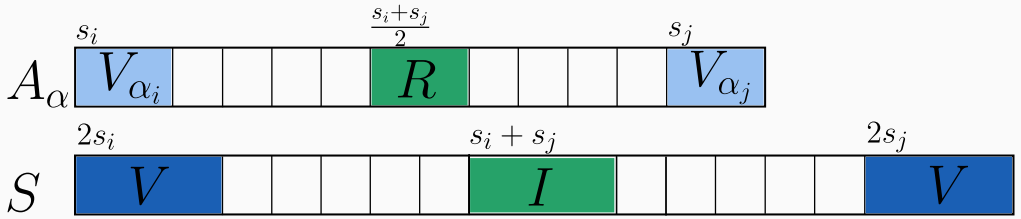

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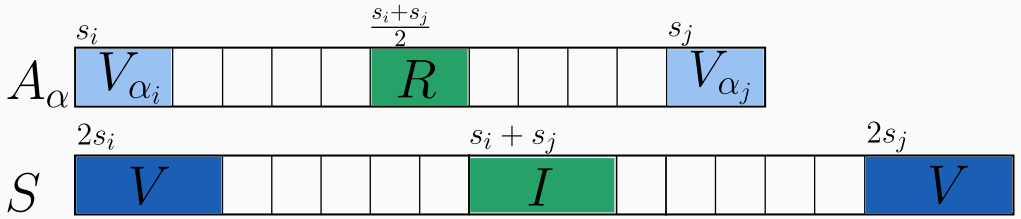

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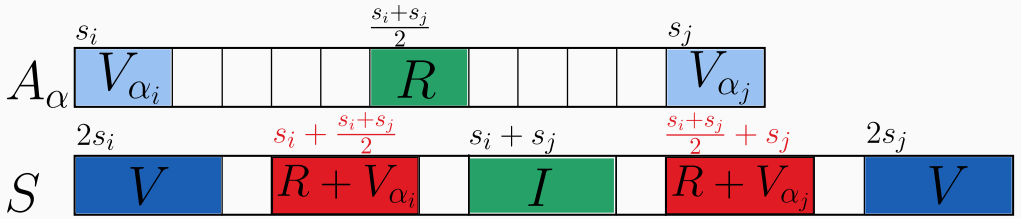

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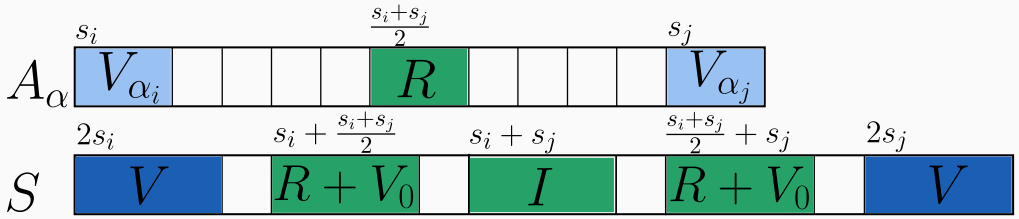

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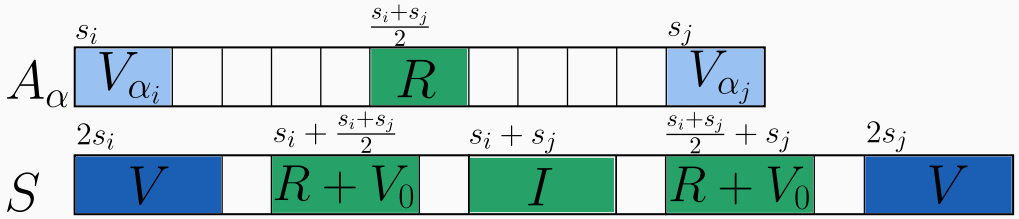

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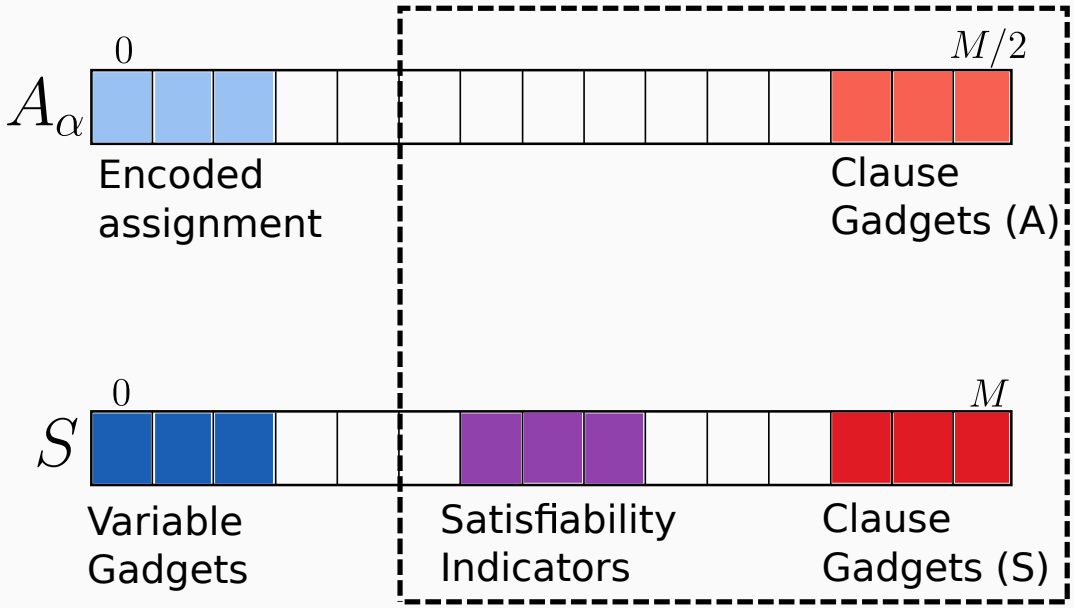
Masking (informally)

Whenever our design of S and A_α is such that $A_\alpha + A_\alpha$ also contains elements not oblivious to α, we can employ masking to resolve it.

Clause Gadgets



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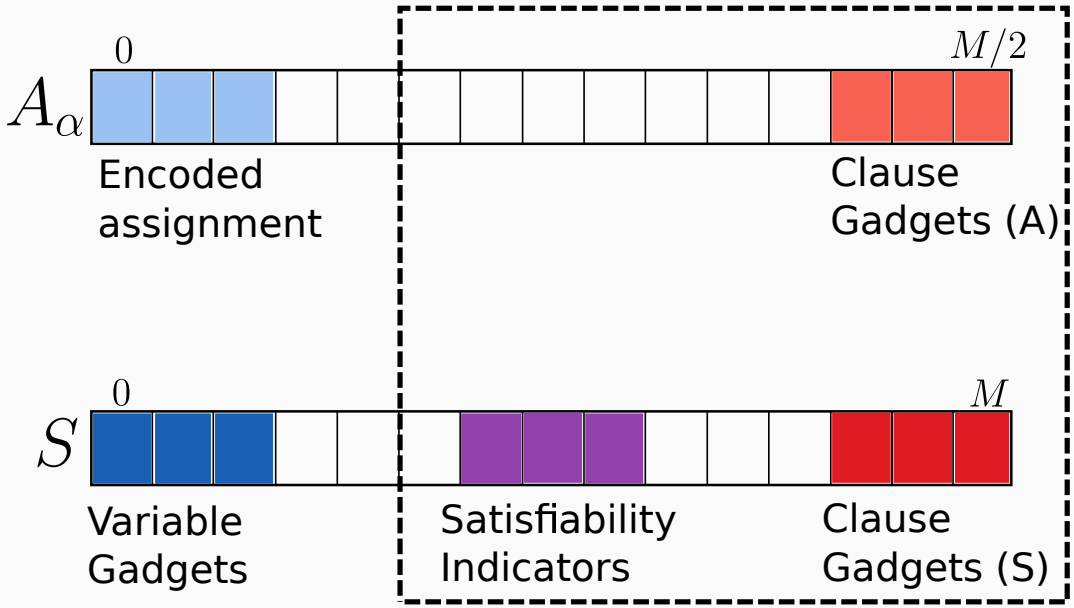

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Remark: To enforce G_k into A_α, let S contain $G_k + G_k$.

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Example (The gadget G_k)

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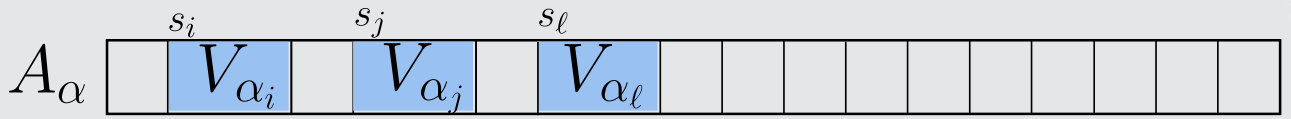

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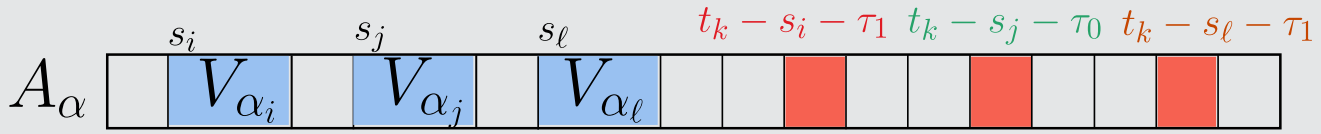

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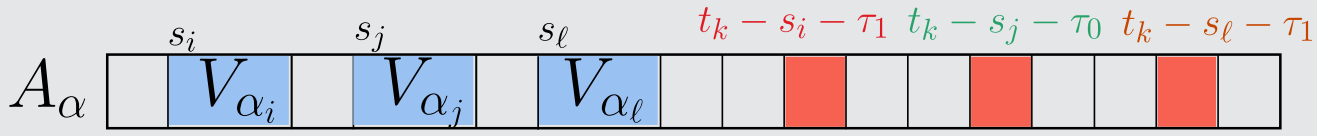

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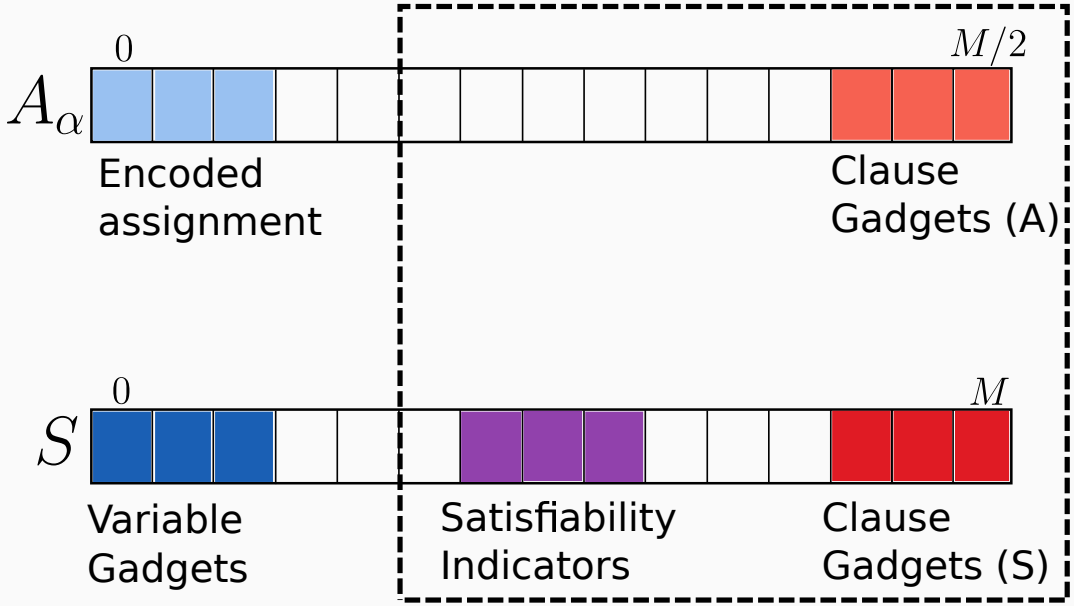
Observation

$A_\alpha + A_\alpha$ contains the set $(s_i + V_{\alpha_i}) + (t_k - s_i - \tau_1)$.

- s_i and $-s_i$ cancel out.
- $-\tau_1$ cancels if and only if $V_{\alpha_i} = V_1$, i.e., if and only if α_i satisfies the clause.

Hence, t_k is in the set if and only if α satisfies the clause.

Clause Gadgets



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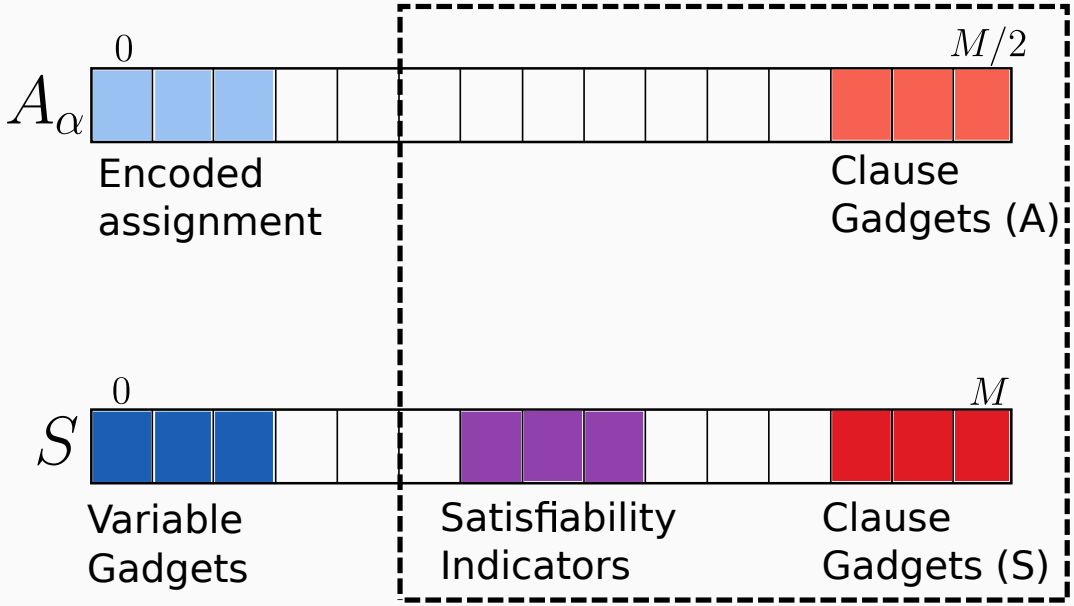

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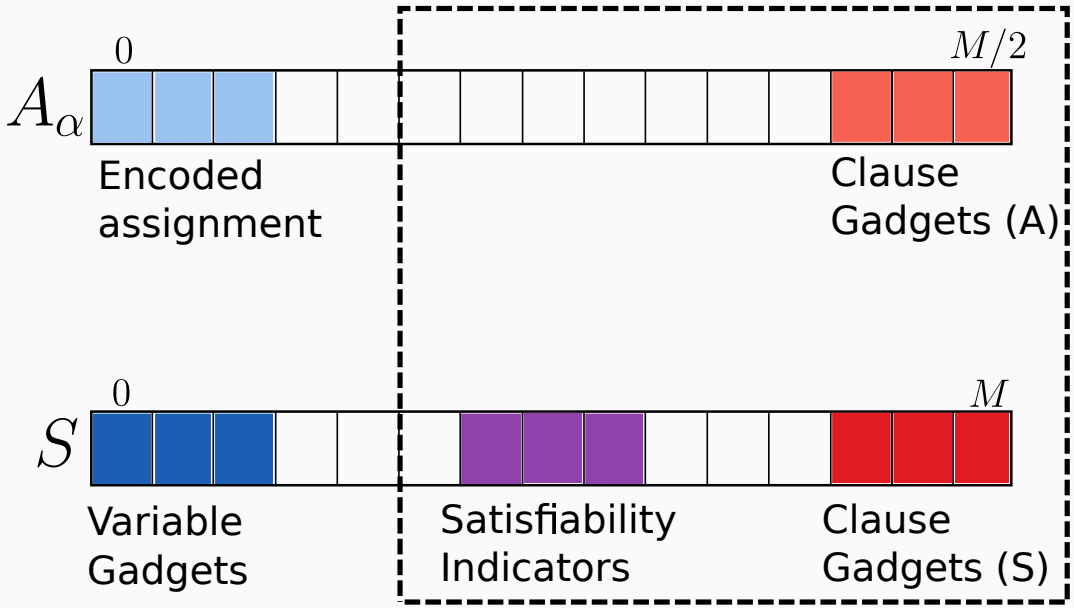

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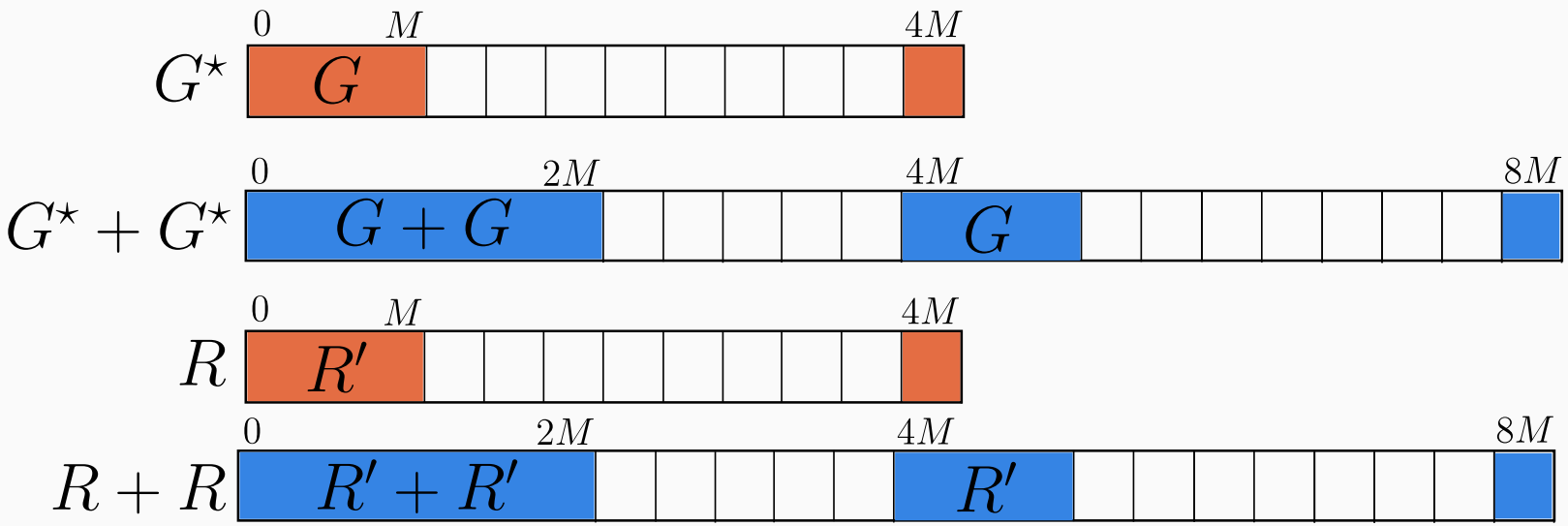
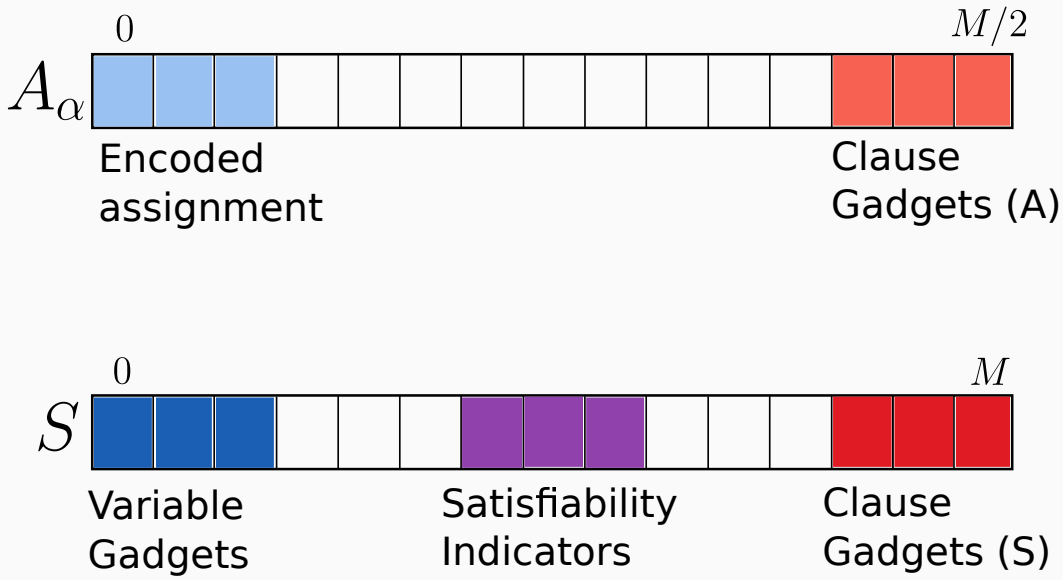


Masking and Positioning, Generalized



Our actual construction is more modular and uses generalized masking and positioning lemmas:

- Masking: Can ignore by-products that depend on α in $A_\alpha + A_\alpha$.
- Positioning: Can ignore solutions $S = A + A$ for $A \notin \{A_\alpha\}_{\alpha \in \{0,1\}^n}$.

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Closing the gap between $2^{\Omega(n^{1/4})}$ and $2^{O(n)}$?

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- What about the second condition?

Graphical Sidon Sets

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What if G is bounded degree? Can we still get $S \subseteq [0, n^{1+o(1)}]$?

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The end!